ficient is insensitive to the algebraic sign of a_1 (see [5]), although positive and negative values of a_1 define different refractive index profiles. The discussion in Section II shows that linear terms cannot be accommodated and therefore this class of profiles must be excluded from those that can be analyzed by the evanescent wave method. The same behavior with respect to a_1 occurs for the slab waveguide but here the different profiles corresponding to positive and negative a_1 are merely reflections of one another about the waveguide axis and therefore have the same modal propagation coefficient. Thus, as shown in [6], the asymmetric slab waveguide is included within the framework of the evanescent wave theory.

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Excitation of Surface Waves and the Scattered Radiation Fields by Rough Surfaces of Arbitrary Slope

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***Abstract*—Surface waves as well as lateral waves are excited when a rough surface is illuminated by the radiation fields. In view of shadowing, these terms of the complete field expansions contribute significantly to the total fields when the transmitter or receiver are near the rough surface. In this work explicit expressions are derived for the coupling between the radiation fields and the surface waves which are guided at the irregular interface between two media. In the analysis, the slope of the rough surface is not restricted and the solutions for both the horizontally and vertically polarized waves are shown to satisfy reciprocity and duality relationships in electromagnetic theory. Special consideration is given to Brewster angles of incidence and scatter and stationary phase techniques. The full-wave solutions are also applied to random and periodic rough surfaces.**

I. Introduction

USING A full-wave approach that accounts for shadowing, it has been shown that the radiation fields scattered from rough surfaces vanish in a continuous manner as the observer moves into the shadow region [4]. Thus when the transmitter or receiver are near the rough boundary, the major contributions to the total fields come from the surface wave and the lateral wave terms of the complete field expansions [1], [2].

In this paper the full-wave approach is used to determine the excitation of the surface wave when the rough surface is illuminated by the radiation field. In addition the scattered radiation fields excited by an incident surface wave are determined. The Kirchoff approach or the Rayleigh hypothesis for instance, cannot be used to solve this problem [3], [6]. Both vertically and horizontally polarized waves are considered and the solutions are shown to satisfy duality and reciprocity relationships in electromagnetic theory.

For the convenience of the reader, the principal elements of the full-wave approach, including the complete expansions of the fields, the exact boundary conditions and the rigorous set of coupled differential equations for the wave amplitudes (generalized telegraphist's equations) are summarized in Section II. In addition explicit expressions for the coupling coefficients are provided.

In Section III second-order iterative solutions for the scattered fields are presented. To remove the small slope restriction inherent in the iterative solutions (while at the same time retaining the relatively simple form of these

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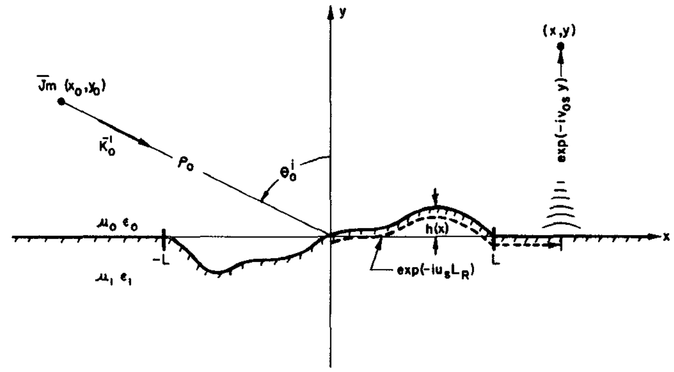

Fig. 1. The scattered surface wave due to incident plane waves.

solutions) a transformation into a variable local coordinate system, that conforms with the slope of the rough surface, is introduced. Thus, the expression for the full-wave solutions derived in this section are shown to be invariant to coordinate transformations. Since upward and downward scattering is accounted for in this analysis, shadowing effects are also considered here (Section IV).

Stationary phase conditions and coupling near the Brewster angle are considered in Section V. Application of the full-wave solutions to random and periodic rough surfaces are given in Section VI.

II. Formulation of the Problem

For horizontally stratified media the electromagnetic fields can be expanded completely in terms of the radiation term, the lateral and the surface waves of the media [1], [7]. When the transmitter or receiver are far from the irregular interface between two semi-infinite media for instance, the scattered radiation field is the dominant term in the full-wave expansion of the field. However, in shadow regions, the radiation fields scattered by rough surfaces vanish [4], and the principal contributions to the scattered fields are due to coupling between the radiation fields and the lateral and surface waves that are guided at the irregular interface between the two media [3]. The contributions of the surface and the lateral waves are also very significant when the transmitter and receiver are just below the earth's surface. Coupling of electromagnetic fields into and out of dielectric waveguides can also be enhanced by a careful study of the coupling mechanism between the radiation fields and the guided waves of the dielectric structure at an irregularly shaped interface.

In this work, a full-wave method is used to determine both the excitation of the surface waves by rough surfaces that are illuminated by the radiation fields as well as the radiation fields scattered by rough surfaces that are excited by surface waves. Using a local coordinate system that varies with the rough surface boundary, restrictions of the earlier iterative solutions are removed [3]. It is assumed that in general, both the permittivity ϵ_m and the permeability μ_m are different for the media $(m=0,1)$ above and below the irregular interface $y=h(x)$ (see Fig. 1), and both horizontally and vertically polarized waves are considered in this work.

For simplicity, the interface $y=h(x)$, the permittivity ϵ_m, and permeability μ_m as well as the z-directed line sources are assumed to be independent of the z axis. Thus, the problem is two-dimensional and the scattered waves are not depolarized by the rough surface. The time dependence, $\exp(i\omega t)$, is factored out throughout this work and the horizontally and vertically polarized waves are assumed to be excited by z-directed electric and magnetic line sources, respectively.

To assist the engineer who is not very familiar with the full-wave approach used in this work, the principal steps in the derivations are given and the solutions are cast in a form that can be directly used to obtain numerical solutions [3].

The principal features of the full-wave approach are: a) use of a complete expansion of the fields; b) imposition of the exact boundary conditions and the irregular interface; c) conversion of Maxwell's equations into a set of generalized telegraphist's equations for the wave amplitudes; d) use of rigorous mathematical procedures (thus, to avoid term-by-term differentiation of the complete field expansions, since the boundary $(y=h(x))$ between the two media is irregular, use is made of Green's theorems); e) the solutions for the scattered fields are cast in a form that can be directly used by the engineer and are simple to interpret physically; f) use is made of a variable coordinate system that conforms with the local features of the rough surface. Thus the solutions are in a form that is invariant under coordinate transformation and earlier restrictions on the slope of the rough surface are removed.

The transverse (y, z) field components for the vertically polarized waves (denoted by the superscript V) are expressed completely in terms of the radiation term (integration with respect to the transform variable v_0) the lateral wave (integration with respect to v_1) and the surface wave term (denoted by the subscript s). Thus [1]

$$\begin{aligned} H_z(x,y) &= H_0^V(x,y)+H_1^V(x,y)+H_s^V(x,y) \\ &\equiv \sum_n H_n^V(x,v)\psi_n^V(v,y) \\ &= \int_0^\infty H_0^V(x,v)\psi_0^V(v,y)\,dv_0+\int_0^\infty H_1^V(x,v)\psi_1^V(v,y)\,dv_1 \\ &\quad + H_s^V(x,v)\psi_s^V(v,y) \end{aligned} \tag{2.1a}$$

and

$$E_y(x,v)=\sum_n E_n^V(x,v)Z^V(v,y)\psi_n^V(v,y) \tag{2.1b}$$

where the field transforms $H_n^V(x,v)$ and $E_n^V(x,v)$ are for $n=0,1,s$

$$H_n(x,v)=\int_{-\infty}^{\infty} H_z(x,y)Z^V(v,y)N_n^V(v)\psi_n^V(v,y)\,dy \tag{2.2a}$$

and

$$E_n^V(x,v)=\int_{-\infty}^{\infty} E_y(x,v)N_n^V(v)\psi_n^V(v,y)\,dy. \tag{2.2b}$$

Similarly, the complete expansions for the horizontally polarized waves (denoted by the superscript H) are

$$E_z(x,y)=\sum_n E_n^H(x,v)\psi_n^H(v,y) \quad (2.3a)$$

and

$$H_y(x,y)=\sum_n H_n^H(x,v)Y^H(v,y)\psi_n^H(v,y) \quad (2.3b)$$

in which the field transforms $E_n^H(x,v)$ and $H_n^H(x,v)$ are for $n=0,1,s$

$$E_n^H(x,v)=\int_{-\infty}^{\infty} E_z(x,y)Y_n^H(v,y)N_n^H(v)\psi_n^H(v,y)\,dy \quad (2.4a)$$

$$H_n^H(x,v)=\int_{-\infty}^{\infty} H_y(x,y)N_n^H(v)\psi_n^H(v,y)\,dy. \quad (2.4b)$$

In the above expressions the symbol Σ_n denotes summation (integration) over the complete wave spectrum consisting of the radiation fields, the lateral waves, and the surface waves. The basis functions for the radiation, lateral and surface wave terms are, respectively, ψ_0^P, ψ_1^P and $\psi_s^P(P=V,H)$ [5]. The normalization coefficient for $n=s$ is

$$N_s^P(v)=1, \qquad P=V,H \quad (2.5a)$$

and for $n=1,2$

$$N_n^P=R_n^{Ph}(v)/2\pi I_n^P(v), \qquad P=V,H \quad (2.5b)$$

in which the transverse wave impedance and admittances are

$$I_n^P(v,y)=\begin{cases} Z_n^V(v,y)=\dfrac{u}{\omega\epsilon_n}=\begin{cases}\dfrac{u}{\omega\epsilon_0}=Z_0^V(v)\\[2mm] \dfrac{u}{\omega\epsilon_1}=Z_1^V(v)\end{cases} & (2.6a)\\[6mm] Y_n^H(v,y)=\dfrac{u}{\omega\mu_n}=\begin{cases}\dfrac{u}{\omega\mu_0}=Y_0^H(v)\\[2mm] \dfrac{u}{\omega\mu_1}=Y_1^H(v)\end{cases} & (2.6b)\end{cases}$$

The reflection coefficients with respect to the reference surface $y=0$ are for $P=V,H$

$$R_0^{Ph}(v)=R_0^P(v)\exp(i2v_0h),\ R_1^{Ph}=R_1^P(v)\exp(-i2v_0h). \quad (2.7)$$

The Fresnel reflection coefficients, $R_0^P(v)$, are

$$R_0^V(v)=-R_1^V(v)=\frac{v_0\epsilon_1-v_1\epsilon_0}{v_0\epsilon_1+v_1\epsilon_0}=\frac{\eta_0C_0-\eta_1C_1}{\eta_0C_0+\eta_1C_1} \quad (2.8a)$$

and

$$R_0^H(v)=-R_1^H(v)=\frac{v_0\mu_1-v_1\mu_0}{v_0\mu_1+v_1\mu_0}=\frac{\eta_1C_0-\eta_0C_1}{\eta_1C_0+\eta_0C_1}. \quad (2.8b)$$

The modal equation for the surface waves is

$$1/R_0^P(v_s)=0. \quad (2.9)$$

The intrinsic impedance for medium m is $\eta_m=(\mu_m/\epsilon_m)^{1/2}$, C_0 and C_1 are the cosines of the angles between the y axis and the wave normals in medium $m=0$ $(y>h(x))$ and medium $m=1$ $(y<h(x))$, respectively, (see Fig. 1)

$$v_m=k_m\cos\theta_m=k_mC_m, \qquad m=0,1 \quad (2.10a)$$

in which $k_m=\omega(\mu_m\epsilon_m)^{1/2}$, is the wavenumber for medium m. The vector wavenumbers for the incident and the scattered waves are, respectively,

$$\bar{k}_m^i=u^i\bar{a}_x+v_m^i\bar{a}_y=k_m^i(S_m^i\bar{a}_x-C_m^i\bar{a}_y)=k_m^i\bar{n}_m^i, \qquad m=0,1 \quad (2.10b)$$

and

$$\bar{k}_m^f=u^f\bar{a}_x+v_m^f\bar{a}_y=k_m^f(S_m^f\bar{a}_x+C_m^f\bar{a}_y)=k_m^f\bar{n}_m^f, \qquad m=0,1 \quad (2.10c)$$

in which C_m^i, C_m^f, and S_m^i, S_m^f are the cosines and sines of the angles between the wave normals and the y axis for the incident and scattered waves and $\bar{n}_m^i$ and $\bar{n}_m^f$ are unit vectors in the directions of the wave normals in medium $m=0,1$.

Using the full-wave approach, the field transforms E_n^P and H_n^P are expressed in terms of the forward and backward wave amplitudes

$$H_n^P=a_n^P\pm b_n^P,\ E_n^P=a_n^P\mp b_n^P, \qquad P=V,H, \qquad n=0,1,s \quad (2.11)$$

where the upper and lower sines are for $P=V$ and H, respectively. The generalized telegraphist's equations are obtained by substituting the complete field equations (2.1)–(2.4) into Maxwell's equation for the transverse field components, and using the orthogonality relationships between the basis functions. Use is made of Green's theorems to avoid term-by-term differentiation of the complete field expansions and the following exact boundary conditions are imposed at the irregular boundary $y=h(x)$ between the two media $(m=0,1)$ for vertically polarized waves

$$\left[E_y\frac{dh}{dx}+\frac{1}{i\omega\epsilon}\frac{\partial}{\partial y}H_z\right]_{h^-}^{h^+}=0, \qquad [H_z]_{h^-}^{h^+}=0 \quad (2.12a)$$

and for horizontally polarized waves

$$\left[H_y\frac{dh}{dx}-\frac{1}{i\omega\mu}\frac{\partial}{\partial y}E_z\right]_{h^-}^{h^+}=0, \qquad [E_z]_{h^-}^{h^+}=0. \quad (2.12b)$$

Thus Maxwell's equations are transformed into the following rigorous set of differential equations for the wave amplitudes [2].

$$-\frac{d}{dx}a_n^P(x,v)-iua_n^P(x,v)=\sum_{n'} S_{PP}^{BA}(n,n',v,v')a_{n'}^P(x,v')+S_{PP}^{BB}(n,n',v,v')b_{n'}^P(x,v)+J_n^P/2 \quad (2.13a)$$

$$-\frac{d}{dx}b_n^P(x,v)+iub_n^P(x,v)=\sum_{n'} S_{PP}^{AA}(n,n',v,v')a_{n'}^P(x,v')+S_{PP}^{AB}(n,n',v,v')b_{n'}^P(x,v)-J_n^P/2 \quad (2.13b)$$

in which the line source transforms are

$$J_n^P(x,v)=\int_{-\infty}^{\infty} J_l(x,y)N_n^P(v)\psi_n^P(v,y)\,dy, \qquad \begin{cases} l=e \text{ for } P=H\\ l=m \text{ for } P=V\end{cases}. \quad (2.14)$$

The transmission and reflection scattering coefficients are, respectively,

$$S_{PP}^{BA}(n,n',v,v')=S_{PP}^{AB}(n,n',v,v')$$
$$=\frac{1}{2}\left[\frac{N_n^P(v)}{N_{n'}^P(v')}G^P(n',n,v',v)-G^P(n,n',v,v')\right] \quad (2.15a)$$

and

$$S_{PP}^{AA}(n,n',v,v')=S_{PP}^{BB}(n,n',v,v')$$
$$=\frac{1}{2}\left[\frac{N_n^P(v)}{N_{n'}^P(v')}G^P(n',n,v',v)+G^P(n,n',v,v')\right] \quad (2.15b)$$

in which

$$G^P(n,n',v,v')=\left[\frac{I^P(v',y)N_n^P(v)}{v^2-v'^2}\cdot\left\{\psi_{n'}^P(v',y)\frac{\partial^2}{\partial x\partial y}\psi_n^P(v,y)-\frac{\partial}{\partial y}\psi_{n'}^P(v',y)\frac{\partial}{\partial x}\psi_n^P(v,y)\right\}\right]_{h^-}^{h^+}. \quad (2.16)$$

Using the differential equations and boundary conditions for the basis functions $\psi_n^P(v,y)$, it can be shown that

$$\frac{\partial}{\partial x}\psi_n^P(v,h)=\left[\frac{\partial}{\partial x}\psi_n^P(v,y)+\frac{\partial}{\partial y}\psi_n^P(v,y)\frac{dh}{dx}\right]_{y=h} \quad (2.17a)$$

and

$$\frac{\partial^2}{\partial x\partial y}\psi_n^P(v,h)=\left[\frac{\partial^2}{\partial x\partial y}\psi_n^P(v,y)+\frac{\partial^2}{\partial y^2}\psi_n^P(v,y)\frac{dh}{dx}\right]_{y=h}. \quad (2.17b)$$

Thus it follows that

$$G^P(n,n',v,v')=\left[\frac{I(',y)N_n^P(v)}{v^2-v'^2}\left\{v^2\psi_{n'}^P(v',y)\psi_n^P(v,y)+\frac{\partial}{\partial y}\psi_{n'}^P(v',y)\frac{\partial}{\partial y}\psi_n^P(v,y)\right\}\right]_{h^-}^{h^+}\frac{dh}{dx}. \quad (2.18)$$

In this work coupling between the radiation fields and the guided surface waves are investigated in detail, thus for $n,n'=0,s$

$$G^P(s,0,v_s,v')=\frac{\psi_0^P(v',h)\psi_s^P(v_s,h)}{v_s^2-v'^2}\cdot\left[I_0^P(v')(v_{0s}^2-v_{0s}v_1')-I_1^P(v')(v_{1s}^2-v_{0s}v_1')\right]\frac{dh}{dx} \quad (2.19a)$$

and

$$\frac{N_s^P(v)}{N_0^P(v')}G^P(0,s,v',v_s)=\frac{\psi_0^P(v',h)\psi_s^P(v_s,h)}{v'^2-v_s^2}\cdot\left[I_0^P(v_s)(v_0'^2-v_{0s}v_1')-I_1^P(v_s)(v_1'^2-v_{0s}v_1')\right]\frac{dh}{dx}. \quad (2.19b)$$

Hence for vertically polarized waves

$$S_{VV}^{BA}(s,0,v_s,v')=\frac{\psi_0^V(v',h)\psi_s^V(v_s,h)}{2(u_s-u')\omega\epsilon_0}\cdot\left[(u'u_s+v_1'v_{0s})\left(\frac{1}{\epsilon_r}-1\right)+k_0^2(1-\mu_r)\right]\frac{dh}{dx} \quad (2.20a)$$

and for horizontally polarized waves

$$S_{HH}^{BA}(s,0,v_s,v')=\frac{\psi_0^H(v',h)\psi_s^H(v_s,h)}{2(u_s-u')\omega\mu}\cdot\left[(u'u_s+v_1'v_{0s})\left(\frac{1}{\mu_r}-1\right)+k_0^2(1-\epsilon_r)\right]\frac{dh}{dx} \quad (2.20b)$$

in which $\epsilon_r=\epsilon_1/\epsilon_0$ and $\mu_r=\mu_1/\epsilon_0$ are the relative permittivity and permeability, respectively.

The electric and magnetic line sources of intensities I_e (amps) and I_m (volts) located at (x_0,y_0) are expressed in terms of the Dirac delta functions

$$\bar{J}=J_l\bar{a}_z=I_l\delta(x-x_0)\delta(y-y_0)\bar{a}_z, \qquad l=e,m. \quad (2.21a)$$

Thus the source transforms (2.14) are

$$J_n^P=I_l\delta(x-x_0)N_n^P(v)\psi_n^P(v,y_0), \qquad l=e,m \text{ for } P=H,V. \quad (2.21b)$$

III. Coupling Between the Radiation Fields and the Surface Waves When the Slope of the Rough Surface is Small-Iterative Solutions

To obtain the first-order iterative solutions for the generalized telegraphist's equations, the transmission and reflection scattering coefficients are ignored in (2.13). Thus, the radiation field due to a magnetic line source for the unperturbed case $h=$const. is given by [1]

$$H_0(x,y)=\frac{I_m i\omega\epsilon_0}{2(2\pi i k_0\rho_d)^{1/2}}\cdot\left[\exp(-ik_0\rho_d)+\left(\frac{\rho_d}{D}\right)^{1/2}R_0^V(v')\exp(-ik_0D)\right] \quad (3.1a)$$

in which the first term is the direct wave and the second is the wave specularly reflected at the angle θ_0^i. Thus in (3.1a)

$$v_0^i=k_0\cos\theta_0^i \quad (3.1b)$$

and

$$\rho_d=\left[(x-x_0)^2+(y-y_0)^2\right]^{1/2}$$
$$D=\left[(x+x_0)^2+(y-y_0)^2\right]^{1/2}. \quad (3.1c)$$

The incident wave at the origin is

$$H_0^i=\frac{I_m i\omega\epsilon_0}{2(2\pi i k_0\rho_0)^{1/2}}\exp(-ik_0\rho_0) \quad (3.2a)$$

where

$$\rho_0=(x_0^2+y_0^2)^{1/2}. \quad (3.2b)$$

Similarly, the unperturbed surface wave for $y>y_0$ is given by [1]

$$H_s(x,y)=-\frac{I_m v_{0s} i\omega\epsilon_0}{u_s[1-1/\epsilon_r^2]}\cdot\exp[-iu_s(x-x_0)]\exp[-iv_{0s}(y+y_0)] \quad (3.3)$$

in which v_{0s} and v_{1s} are derived from the modal equation (2.9). Thus

$$v_{0s}=k_0C_{0s}=-k_0(n_r^2/\epsilon_r^2-1/\epsilon_r^2)^{1/2}/(1-1/\epsilon_r^2)^{1/2} \quad (3.4a)$$

$$v_{1s}=k_1C_{1s}=-v_{0s}\epsilon_r \quad (3.4b)$$

and

$$u_s=k_0S_{0s}=k_0(1-n_r^2/\epsilon_r^2)^{1/2}/(1-1/\epsilon_r^2)^{1/2} \quad (3.4c)$$

where n_r is the relative refractive index $n_r=(\epsilon_r\mu_r)^{1/2}$. The second-order iterative solutions for the wave amplitudes are obtained on substituting the first-order solutions for the wave amplitudes on the right-hand side of (2.13) and solving the resulting equations. These solutions for the wave amplitudes are substituted back into the complete field expansions (2.1)–(2.4) to obtain the second-order iterative solutions for the scattered fields. Thus the surface wave due to the illumination of the rough surface

$$y=h(x), \qquad -L\leqslant x\leqslant L \quad (3.5)$$

by the incident radiation field (3.2) is

$$H_{s0}=G_{s0}^{VV}F_{s0}^{VV}(\theta_{0s},\theta_0^i)I(\theta_{0s},\theta_0^i,h,L) \quad (3.6)$$

where

$$G_{s0}^{VV}=-\frac{2I_m i\omega\epsilon_0\exp(-ik_0\rho_0)}{(2\pi ik_0\rho_0)^{1/2}}\frac{v_{0s}(iv_{0s}2L)}{u_s(1-1/\epsilon_r^2)}\cdot\exp\left[-i\int_0^x u_s\,dx'\right]\exp(-iv_{0s}y) \quad (3.7)$$

and

$$F_{s0}^{VV}(\theta_{0s},\theta_0^i)=\frac{(1-1/\epsilon_r)[\mu_rC_{1s}C_1^i-S_{0s}S_0^i]+(1-\mu_r)}{2C_{0s}(C_0^i+\eta_rC_1^i)}. \quad (3.8)$$

The relative intrinsic impedance is $\eta_r=\eta_1/\eta_0$ and

$$I(\theta_{0s},\theta_0^i,h,L)=\frac{1}{2L}\int_{-L}^{L}\exp\left[ik_0\left(C_0^ih-S_0^ix+\int_0^x S_{0s}\,dx'\right)\right]dx. \quad (3.9)$$

The coefficient G_{s0}^{VV} can be identified with the incident radiation field (3.2) and the outgoing surface wave (3.3). The coupling mechanism is represented by the coefficient F_{s0}^{VV} and the integral I (3.9) over the rough surface $h(x)$. For the reciprocal problem in which the rough surface is excited by a back travelling surface wave, the scattered radiation field in the direction θ_0^f is given by

$$H_{0s}=G_{0s}^{VV}F_{0s}^{VV}(\theta_0^f,-\theta_{0s})I(\theta_{01}^f-\theta_{0s},h,L) \quad (3.10)$$

in which

$$G_{0s}^{VV}=-\frac{-2I_m i\omega\epsilon_0\exp(-ik_0\rho)}{(2\pi ik_0\rho)^{1/2}}\frac{v_{0s}(iv_{0s}2L)}{u_s(1-1/\epsilon_r^2)}\cdot\exp\left(-i\int_0^{x_0}u_s\,dx\right)\exp(iv_{0s}y_0) \quad (3.11)$$

$$F_{0s}^{VV}(\theta_0^f,-\theta_{0s})=F_{s0}^{VV}(\theta_{s0},-\theta_0^f) \quad (3.12)$$

and

$$I(\theta_0^f,-\theta_{0s},h,L)=\frac{1}{2L}\int_{-L}^{L}\left[ik_0\left(C_0^fh+S_0^fx+\int_0^x S_{0s}\,dx'\right)\right]dx. \quad (3.13)$$

The coefficient G_{0s}^{VV} is identified with the incident surface wave and the scattered radiation field at a distance

$$\rho=(x^2+y^2)^{1/2}. \quad (3.14)$$

Thus consistent with reciprocity (3.10) can be obtained from (3.6) on making the following substitutions:

$$\rho_0\to\rho,\quad x\to x_0,\quad y\to y_0,\quad \theta_0^i\to-\theta_0^f. \quad (3.15)$$

The corresponding solutions to the problem of coupling between the radiation field and the surface wave for horizontally polarized waves can be obtained directly from the above results for vertically polarized waves by invoking the duality relationships in electromagnetic theory. Thus, the scattered surface wave excited by the illumination of the rough surface by the incident horizontally polarized radiation field is

$$E_{s0}=G_{s0}^{HH}F_{s0}^{HH}(\theta_{0s},\theta_0^i)I(\theta_{0s},\theta_0^i,h,L) \quad (3.16)$$

in which G_{s0}^{HH} and F_{s0}^{HH} are obtained from the expressions for G_{s0}^{VV} (3.7) and F_{s0}^{VV} (3.8) through the following transformations:

$$I_m\to-I_e,\quad \bar{H}\to-E,\quad \mu\to\epsilon,\quad \epsilon\to\mu,\quad \eta_r\to1/\eta_r. \quad (3.17)$$

Similarly, the scattered horizontally polarized radiation field excited by a backward travelling surface wave incident on the rough surface is given by

$$E_{0s}=G_{0s}^{HH}F_{0s}^{HH}(\theta_0^f,-\theta_{0s})I(\theta_0^f,-\theta_{0s},h,L). \quad (3.18)$$

The expressions for G_{0s}^{HH}, F_{0s}^{HH}, and I in (3.18) are related to G_{0s}^{VV}, F_{0s}^{VV}, and I in (3.10) through the duality relationships (3.17).

IV. Full Wave Solutions When the Slope of the Rough Surface is Large

To remove the small slope assumption introduced to obtain the iterative solutions presented in the previous section and in order to retain the relatively simple form of these solutions, a variable coordinate system that conforms with the local features of the rough surface is used [4]. Thus the rough surface is regarded as a continuum of elementary inclined strips of varying slope and height rather than a continuum of elementary horizontal strips of varying height. The contribution to the total scattered surface wave from an elementary horizontal strip at x, y of width dx' is (3.6) of width dx' is (3.6)

$$dH_{s0}=G_{s0}^{VV}F_{s0}^{VV}(\theta_{0s},\theta_0^i)\exp\{i\phi^i(x')\}\frac{dx'}{2L}\equiv G_{s0}^{VV}F_{s0}^{VV}\,dI(x') \tag{4.1a}$$

in which

$$\phi^i(x')=k_0\left[C_0^i h-S_0^i x'+\int_0^{x'}S_{0s}\,dx''\right]. \tag{4.1b}$$

The corresponding expression for dH_{s0} due to scattering by an inclined strip at (x,y) of length

$$dl=(dx^2+dy^2)^{1/2}=(1+(h')^2)^{1/2}\,dx=dx/\cos\gamma \tag{4.2a}$$

and gradient

$$h'=dh/dx=\tan\gamma \tag{4.2b}$$

is given by performing the following coordinate transformation into the local (variable) coordinates ξ,η (the η axis is normal to the local tangent plane).

$$\xi=x\cos\gamma+y\sin\gamma \tag{4.3a}$$

$$\eta=-x\sin\gamma+y\cos\gamma \tag{4.3b}$$

The incident and scatter angles with respect to the reference plane $y=0$, θ_0^i,θ_0^f are replaced by the local incident and scatter angles with respect to the local tangent plane, thus

$$\theta_0^i\to\theta_0^{i\gamma}=\theta_0^i-\gamma$$
$$\theta_0^f\to\theta_0^{f\gamma}=\theta_0^f+\gamma. \tag{4.4a}$$

The corresponding angles for medium 1 $(y<h)\theta_1^{i\gamma}$ and $\theta_1^{f\gamma}$ are given by Snell's law

$$k_1\sin\theta_1^{i\gamma}=k_0\sin\theta_0^{i\gamma}$$
$$k_1\sin\theta_1^{f\gamma}=k_0\sin\theta_0^{f\gamma}. \tag{4.4b}$$

Under the transformation (4.3) and (4.4)

$$C_0^{i\gamma}\eta-S_0^{i\gamma}\xi=C_0^i h-S_0^i x \tag{4.5}$$

$$u_sL_s(x')=\int_0^{x'}k_0S_{0s}\,dx''/\cos\gamma \tag{4.6a}$$

$$u_sL_R(x)=\int_0^{x}k_0S_{0s}\,dx''/\cos\gamma\,. \tag{4.6b}$$

Thus

$$\phi^i(x')\to\phi^{i\gamma}(x')=k_0(C_0^i h-S_0^i x')+u_sL_s(x'). \tag{4.7}$$

The total surface wave due to the illumination of the rough surface of arbitrary slope by the radiation field is therefore

$$H_{s0}=\frac{1}{2L}\int_{-L}^{L}G_{s0}^{VV}F_{s0}^{VV}(\theta_{0s},\theta_0^i)\cdot\exp\left[ik_0(C_0^i h-S_0^i x')+iu_sL_s(x')\right]\frac{dx'}{\cos\gamma}\,. \tag{4.8}$$

In the expression (4.8), the path length over the rough surface from the origin to the scattering element at x', $L_s(x')$, is given by (4.6a) and in the expression for G_{s0}^{VV} (3.7), the path length from the origin to the receiver, $L_R(x)$, is given by (4.6b). The expression for the full-wave solution (4.8) is invariant to coordinate transformation. The corresponding expressions for H_{0s}, E_{s0}, E_{0s} are obtained from (3.10), (3.16), and (3.18) in a similar manner. Thus for instance in (3.13)

$$\phi^f(x')=k_0\left[C_0^f h+S_0^f x'+\int_0^{x'}S_{0s}\,dx''\right] \tag{4.9a}$$

is replaced by

$$\phi^{f\gamma}(x')=k_0(C_0^f h+S_0^f x')+u_sL_s(x') \tag{4.9b}$$

where $L_s(x')$ is given by (4.6a). Similarly, in (3.1)

$$\int_0^{x_0}u_s\,dx''\to\int_0^{x_0}u_s\,dx''/\cos\gamma=u_sL_0(x_0) \tag{4.10}$$

where $L_0(x_0)$ is the path length along the rough surface from the source at x_0 to the origin.

The full-wave solutions derived in this section are valid for

$$-\pi/2<\theta_0^{i\gamma}=\theta_0^i-\gamma<\pi/2,\quad -\pi/2<\theta_0^{f\gamma}=\theta_0^{f\gamma}+\gamma<\pi/2 \tag{4.11}$$

thus both upward and downward propagating waves with respect to the reference plane $y=0$ are accounted for in this analysis. Only those regions of the rough surface that are illuminated by the source or visible to the observer contribute to the scattered fields. For plane waves incident at an angle θ_0^i, the shadow region extends from x_1^i to x_2^i, where [4]

$$\tan\gamma(x_1^i)=h'(x_1^i)=-\cot\theta_0^i \tag{4.12a}$$

and

$$x_2^i-x_1^i=\left[h(x_2^i)-h(x_1^i)\right]/\tan\gamma(x_1^i). \tag{4.12b}$$

Similarly, for an observation point in the direction θ_0^f, the region of the rough surface extending from x_1^f to x_2^f is not visible to the observer

$$\tan\gamma(x_1^f)=h'(x_1^f)=\cot\theta_0^f \tag{4.13a}$$

and

$$x_2^f-x_1^f=\left[h(x_2^f)-h(x_1^f)\right]/\tan(x_1^f). \tag{4.13b}$$

V. Stationary Phase Conditions and Coupling at the Brewster Angle

In the expression for the scattered surface wave (4.8) the phase $\theta'^{\gamma}(x)$ is stationary when

$$\frac{d}{dx}\phi^{i\gamma}(x)=k_0\left[(C_0^i h'(x)-S_0^i)+S_{0s}/\cos\gamma\right]\to 0. \tag{5.1}$$

Substitute $h'(x_s)=\tan\gamma_s$ and $S_{0s}=\sin\theta_{0s}$ into (5.1) to get

$$\frac{k_0}{\cos\gamma_s}\left[\sin(\gamma_s-\theta_0^i)+\sin\theta_{0s}\right]\to 0\,. \tag{5.2}$$

Hence

$$\theta_0^{i\gamma_s}\equiv\theta_0^i-\gamma_s\to\theta_{0s}\,. \tag{5.3}$$

Thus, as the local angle of incidence $\theta_0^{i\gamma}$ approaches the Brewster angle $\theta_0^B(R_0^P\to 0, \theta_0^B\to\operatorname{Re}\theta_{0s})$ the phase $\phi^{i\gamma}(x)$

tends to be stationary. However, when the local angle of incidence $\theta_0^{i\gamma}$ (and not θ_0^i) [3] approaches the Brewster angle $F_{s0}^{VV}(\theta_{0s}, \theta_0^{i\gamma}) \to 0$. Thus, the major contributions to the scattered surface wave do not necessarily come from the neighborhood of the points where the phase $\theta^{i\gamma}(x)$ is stationary. It should be pointed out that the Kirchoff approach does not account for coupling between the radiation fields and the surface waves [3]. In a similar way it can be shown that for the scattered radiation fields the phase $\phi^{f\gamma}(x)$ is stationary when

$$\theta_0^{f\gamma_s} = \theta_0^f + \gamma_s \to -\theta_{0s}. \tag{5.4}$$

VI. Random and Periodic Rough Surfaces

In order to determine the statistical average of the scattered surface wave, H_{s0} (4.8), it is necessary to know the distributions of the random functions, $h(x)$ and $h'(x) = \tan\gamma(x)$. However, for a stationary random process $h(x)$ and $h'(x)$ are uncorrelated [6]. Thus, the expected value for H_{s0} is given by

$$\langle H_{s0}^{VV} \rangle = \left\langle \frac{G_{s0}^{VV} F_{s0}^{VV}(\theta_{0s}, \theta_0^i)}{\cos\gamma} \right\rangle \cdot \chi(k_0 C_0^i) \operatorname{sinc}\left[k_0 L(S_0^i - S_{0s})\right] \tag{6.1}$$

in which it is assumed for simplicity that $L_s(x') \simeq x'$. The one-dimensional characteristic function is

$$\chi(k_0 C_0^i) = \int_{-\infty}^{\infty} w(h) \exp(ik_0 C_0^i h)\, dh \tag{6.2}$$

where $w(h)$ is the distribution function for $h(x)$. For slightly rough surfaces, $\gamma \to 0$ and

$$\left\langle \frac{G_{s0}^{VV} F_{s0}^{VV}(\theta_{0s}, \theta_0^{i\gamma})}{\cos\gamma} \right\rangle \to G_{s0}^{VV} F_{s0}^{VV}(\theta_{0s}, \theta_0^i). \tag{6.3}$$

Similarly, the small slope approximation for the variance of H_{0s} is given by

$$D\{H_{s0}\} = |G_{s0}^{VV} F^{VV}(\theta_{0s}, \theta_0^i)|^2 \frac{1}{2L} \cdot \int_{-L}^{L} \exp\left[-ik_0\tau(S_0^i - S_{0s})\right] \cdot \left[\chi_2(k_0 C_0^i, -k_0 C_0^i) - \chi(k_0 C_0^i)\chi^*(k_0 C_0^i)\right] d\tau \tag{6.4}$$

in which the symbol * denotes the complex conjugate and χ_2 is the two-dimensional characteristic function

$$\chi_2(k_0 C_0^i - k_0 C_0^i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(h_1, h_2) \cdot \exp\left[ik_0 C_0^i (h_1 - h_2)\right] dh_1\, dh_1 \tag{6.5}$$

where $W(h_1, h_2)$ is the joint distribution function of $h_1 = h(x_1)$ and $h_2 = h(x_2)$ and $\tau = x_1 - x_2$. The statistical average and the variance for the scattered radiation field due to an incident surface wave H_{0s} are obtained in a similar manner. Thus, consistent with reciprocity they can be obtained directly from (6.1) and (6.4) on making the substitutions (3.15). Similarly, the corresponding expressions for the horizontally polarized waves, E_{s0} and E_{0s}, can be obtained by invoking the duality relationships (3.17).

For an N element periodic rough surface of period $2L$

$$h(x+2L) = h(x), \qquad -NL \leqslant x \leqslant NL \tag{6.6}$$

the scattered surface wave is obtained by multiplying the single element scattered wave $(-L<x<L)$ by the N element array factor

$$F_A(\theta_{0s}, \theta_0^i) = \sin\left[Nk_0(S_0^i L - S_{0s} L_s^L)\right] / \sin\left[k_0(S_0^i L - S_{0s} L_s^L)\right] \tag{6.7}$$

where

$$L_s^L \equiv L_s(L). \tag{6.8}$$

The array factor is maximum for

$$\operatorname{Re} k_0(S_0^i L - S_{0s} L_s^L) = m\pi, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{6.9a}$$

thus

$$S_0^i = \frac{m\lambda}{2L} + \operatorname{Re}(S_{0s} L_s^L / L). \tag{6.9b}$$

The array factor for the scattered radiation field H_{0s} is given by

$$F_A(\theta_0^f, -\theta_{0s}) = F_A(\theta_{0s}, \theta_0^i = -\theta_0^f). \tag{6.10}$$

Thus it can be obtained from (6.7) by replacing S_0^i by $-S_0^f$. The scattered radiation fields are therefore maximum for

$$S_0^f = \frac{m\lambda}{2L} - \operatorname{Re}(S_{0s} L_s^L / L). \tag{6.11}$$

VII. Concluding Remarks

The contribution to the total fields from the first, second, and third terms on the right hand side of (2.1) or (2.3) are the radiation field, the lateral and the surface waves, respectively. As the observer moves into the shadow region, the contribution from the first term (the radiation field) vanishes in a continuous manner [4]. Thus, the surface waves and the lateral waves that are guided at the interface between two different media contribute significantly to the total fields when the transmitter or receiver are near the rough interface [3]. In this work the surface wave excited by an incident radiation field as well as the scattered radiation fields excited by an incident surface wave are derived using a full-wave approach [1], [2]. The Kirchoff approach or the Rayleigh hypothesis for instance, cannot be applied to this problem [3], [6].

To remove the earlier restrictions on the slope of the rough surface, [3], a transformation to a variable, local coordinate system has been used. In addition both ϵ and μ are assumed to be different for $y>h$ and $y<h$. Both vertically and horizontally polarized waves are considered here and the results are also applied to random and periodic rough surfaces. The solutions are shown to satisfy duality and reciprocity relations in electromagnetic theory and they are invariant to coordinate transformations. Since the full-wave approach accounts for upward and down-

ward scattering, shadowing effects are also considered in this work.

It is shown that the phase $\phi^{i\gamma}(x)$ in (4.5) is stationary when the local angle of incidence $\theta^{i\gamma}=\theta^{i}-\gamma$ approaches the Brewster angle. However, at this angle, $F^{VV}(\theta_{0s},\theta_0^i)\to 0$. Thus the major contributions to the scattered surface waves H_{s0} do not necessarily come for the neighborhood of the stationary phase points.

The full-wave approach presented here may also be used to determine the coupling of electromagnetic fields into and out of dielectric waveguides with irregular boundaries.

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Comparative Testing of Leaky Coaxial Cables for Communications and Guided Radar

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***Abstract*—Leaky coaxial cables are finding increasing use in communications systems involving mines, tunnels, railroads, and highways, and in new obstacle detection, or guided radar, schemes for ground transportation and perimeter surveillance. This paper describes the theory and operation of a new laboratory testing technique for these leaky cables based on a novel form of cavity resonator. The technique yields highly consistent and repeatable results that usefully assist in the prediction of the performance of full-size systems, from a simple test on a small sample of cable in a laboratory setting.**

I. Introduction

A. Leaky Coaxial Cables

LEAKY COAXIAL cables are generating increasing interest as a means of providing continuous-access guided communications (CAGC) in tunnels and mines, and in guided ground transportation systems [1]–[3]. Many different types are currently being marketed, or tested experimentally, and a selection is shown in Fig. 1, with the designations as used throughout this paper as described in Table I. Also included is conventional twin feeder, Fig. 1(g), to draw attention to the major characteristics shared by all the types illustrated. They are all open electromagnetic waveguides in which the signal energy is guided along a prescribed linear route, with the fields being confined both inside the cable and outside it, within its immediate vicinity, thus enabling signals to be coupled into immediately adjacent mobile communications units.

With the exception of the twin feeder, all these leaky cables are coaxial in form and include a partially open outer conductor.

In all these cases where periodic holes or slots occur, the spacing is very much less than a wavelength and all the cables illustrated act as slow-wave open guiding structures or surface waveguides [4].

B. Guided Radar

A vast amount of work on surface waveguides for railroad communications has been done in Japan and elsewhere over many years and some of the earlier work

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